矩阵、向量求导法则

（1）行向量对元素求导

设 $\mathbf{y}^T = [y_1 \quad \cdots \quad y_n]$ 是 n 维行向量，x 是元素，则 $\frac{\partial \mathbf{y}^T}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} & \cdots & \frac{\partial y_n}{\partial x} \end{bmatrix}$。

（2）列向量对元素求导

设 $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ 是 m 维列向量，x 是元素，则 $\frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \vdots \\ \frac{\partial y_m}{\partial x} \end{bmatrix}$。

（3）矩阵对元素求导

设 $Y = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{m1} & \cdots & y_{mn} \end{bmatrix}$ 是 $m \times n$ 矩阵，x 是元素，则

$$\frac{\partial Y}{\partial x} = \begin{bmatrix} \frac{\partial y_{11}}{\partial x} & \cdots & \frac{\partial y_{1n}}{\partial x} \\ \vdots & & \\ \frac{\partial y_{m1}}{\partial x} & \cdots & \frac{\partial y_{mn}}{\partial x} \end{bmatrix}。$$

（4）元素对行向量求导

设 y 是元素，$\mathbf{x}^T = [x_1 \quad \cdots \quad x_q]$ 是 q 维行向量，则 $\frac{\partial y}{\partial \mathbf{x}^T} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \cdots & \frac{\partial y}{\partial x_q} \end{bmatrix}$。

（5）元素对列向量求导

设 y 是元素，$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ 是 p 维列向量，则 $\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \vdots \\ \frac{\partial y}{\partial x_p} \end{bmatrix}$。

（6）元素对矩阵求导

设 y 是元素，$X = \begin{bmatrix} x_{11} & \cdots & x_{1q} \\ \vdots & & \vdots \\ x_{p1} & \cdots & y_{pq} \end{bmatrix}$ 是 $p \times q$ 矩阵，则

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \frac{\partial y}{\partial x_{11}} & \cdots & \frac{\partial y}{\partial x_{1q}} \\ \vdots & & \\ \frac{\partial y}{\partial x_{p1}} & \cdots & \frac{\partial y}{\partial x_{pq}} \end{bmatrix}。$$

（7）行向量对列向量求导

设 $\mathbf{y}^T=[y_1 \quad \cdots \quad y_n]$ 是 n 维行向量，$\mathbf{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ 是 p 维列向量，则

$$\frac{\partial \mathbf{y}^T}{\partial \mathbf{x}}=\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\ \vdots & & \\ \frac{\partial y_1}{\partial x_p} & \cdots & \frac{\partial y_n}{\partial x_p} \end{bmatrix}。$$

（8）列向量对行向量求导

设 $\mathbf{y}=\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ 是 m 维列向量，$\mathbf{x}^T=[x_1 \quad \cdots \quad x_q]$ 是 q 维行向量，则

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}^T}=\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_q} \\ \vdots & & \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_q} \end{bmatrix}。$$

（9）行向量对行向量求导

设 $\mathbf{y}^T=[y_1 \quad \cdots \quad y_n]$ 是 n 维行向量，$\mathbf{x}^T=[x_1 \quad \cdots \quad x_q]$ 是 q 维行向量，则

$$\frac{\partial \mathbf{y}^T}{\partial \mathbf{x}^T}=\begin{bmatrix} \frac{\partial \mathbf{y}^T}{\partial x_1} & \cdots & \frac{\partial \mathbf{y}^T}{\partial x_q} \end{bmatrix}。$$

（10）列向量对列向量求导

设 $\mathbf{y}=\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ 是 m 维列向量，$\mathbf{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ 是 p 维列向量，则 $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}=\begin{bmatrix} \frac{\partial y_1}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial y_m}{\partial \mathbf{x}} \end{bmatrix}$。

（11）矩阵对行向量求导

设 $Y=\begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{m1} & \cdots & y_{mn} \end{bmatrix}$ 是 $m\times n$ 矩阵，$\mathbf{x}^T=[x_1 \quad \cdots \quad x_q]$ 是 q 维行向量，则

$$\frac{\partial Y}{\partial \mathbf{x}^T}=\begin{bmatrix}\frac{\partial Y}{\partial x_1} & \cdots & \frac{\partial Y}{\partial x_q}\end{bmatrix}。$$

（12）矩阵对列向量求导

设 $Y=\begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{m1} & \cdots & y_{mn}\end{bmatrix}$ 是 $m\times n$ 矩阵，$\mathbf{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_p\end{bmatrix}$ 是 p 维列向量，则

$$\frac{\partial Y}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial y_{11}}{\partial \mathbf{x}} & \cdots & \frac{\partial y_{1n}}{\partial \mathbf{x}} \\ \vdots & & \vdots \\ \frac{\partial y_{m1}}{\partial \mathbf{x}} & \cdots & \frac{\partial y_{mn}}{\partial \mathbf{x}}\end{bmatrix}。$$

（13）行向量对矩阵求导

设 $\mathbf{y}^T=[y_1 \quad \cdots \quad y_n]$ 是 n 维行向量，$X=\begin{bmatrix} x_{11} & \cdots & x_{1q} \\ \vdots & & \vdots \\ x_{p1} & \cdots & y_{pq}\end{bmatrix}$ 是 $p\times q$ 矩阵，则

$$\frac{\partial \mathbf{y}^T}{\partial X}=\begin{bmatrix}\frac{\partial \mathbf{y}^T}{\partial x_{11}} & \cdots & \frac{\partial \mathbf{y}^T}{\partial x_{1q}} \\ \vdots & & \\ \frac{\partial \mathbf{y}^T}{\partial x_{p1}} & \cdots & \frac{\partial \mathbf{y}^T}{\partial x_{pq}}\end{bmatrix}。$$

（14）列向量对矩阵求导

设 $\mathbf{y}=\begin{bmatrix} y_1 \\ \vdots \\ y_m\end{bmatrix}$ 是 m 维列向量，$X=\begin{bmatrix} x_{11} & \cdots & x_{1q} \\ \vdots & & \vdots \\ x_{p1} & \cdots & y_{pq}\end{bmatrix}$ 是 $p\times q$ 矩阵，则

$$\frac{\partial \mathbf{y}}{\partial X}=\begin{bmatrix}\frac{\partial y_1}{\partial X} \\ \vdots \\ \frac{\partial y_m}{\partial X}\end{bmatrix}。$$

（15）矩阵对矩阵求导

设 $Y=\begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{m1} & \cdots & y_{mn}\end{bmatrix}=\begin{bmatrix}\mathbf{y}_1^T \\ \vdots \\ \mathbf{y}_m^T\end{bmatrix}$ 是 $m\times n$ 矩阵，$X=\begin{bmatrix} x_{11} & \cdots & x_{1q} \\ \vdots & & \vdots \\ x_{p1} & \cdots & y_{pq}\end{bmatrix}$

$=[\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_q]$ 是 $p\times q$ 矩阵，则

$$\frac{\partial Y}{\partial X}=\begin{bmatrix}\frac{\partial Y}{\partial \mathbf{x}_1} & \cdots & \frac{\partial Y}{\partial \mathbf{x}_q}\end{bmatrix}=\begin{bmatrix}\frac{\partial \mathbf{y}_1^T}{\partial X}\\ \vdots \\ \frac{\partial \mathbf{y}_m^T}{\partial X}\end{bmatrix}=\begin{bmatrix}\frac{\partial \mathbf{y}_1^T}{\partial \mathbf{x}_1} & \cdots & \frac{\partial \mathbf{y}_1^T}{\partial \mathbf{x}_q}\\ \vdots & & \vdots \\ \frac{\partial \mathbf{y}_m^T}{\partial \mathbf{x}_1} & \cdots & \frac{\partial \mathbf{y}_m^T}{\partial \mathbf{x}_q}\end{bmatrix}。$$

例 设 $\frac{\partial A}{\partial X}=\begin{bmatrix}2xy & y^2 & y\\ x^2 & 2xy & x\end{bmatrix}$，$X=\begin{bmatrix}x\\ y\end{bmatrix}$，根据（12）矩阵对列向量求导法则，有

$$\frac{\partial^2 A}{\partial X^2}=\begin{bmatrix}\frac{\partial(2xy)}{\partial X} & \frac{\partial(y^2)}{\partial X} & \frac{\partial y}{\partial X}\\ \frac{\partial(x^2)}{\partial X} & \frac{\partial(2xy)}{\partial X} & \frac{\partial x}{\partial X}\end{bmatrix}=\begin{bmatrix}2y & 0 & 0\\ 2x & 2y & 1\\ 2x & 2y & 1\\ 0 & 2x & 0\end{bmatrix}。$$

例 设 $Y=\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}$，$X=\begin{bmatrix}u & x\\ v & y\\ w & z\end{bmatrix}$，根据（15）矩阵对矩阵求导法则，有

$$\frac{\partial Y}{\partial X}=\begin{bmatrix}\frac{\partial\begin{bmatrix}a & b & c\end{bmatrix}}{\partial\begin{bmatrix}u\\ v\\ w\end{bmatrix}} & \frac{\partial\begin{bmatrix}a & b & c\end{bmatrix}}{\partial\begin{bmatrix}x\\ y\\ z\end{bmatrix}}\\ \frac{\partial\begin{bmatrix}d & e & f\end{bmatrix}}{\partial\begin{bmatrix}u\\ v\\ w\end{bmatrix}} & \frac{\partial\begin{bmatrix}d & e & f\end{bmatrix}}{\partial\begin{bmatrix}x\\ y\\ z\end{bmatrix}}\end{bmatrix}=\begin{bmatrix}\frac{\partial a}{\partial u} & \frac{\partial b}{\partial u} & \frac{\partial c}{\partial u} & \frac{\partial a}{\partial x} & \frac{\partial b}{\partial x} & \frac{\partial c}{\partial x}\\ \frac{\partial a}{\partial v} & \frac{\partial b}{\partial v} & \frac{\partial c}{\partial v} & \frac{\partial a}{\partial y} & \frac{\partial b}{\partial y} & \frac{\partial c}{\partial y}\\ \frac{\partial a}{\partial w} & \frac{\partial b}{\partial w} & \frac{\partial c}{\partial w} & \frac{\partial a}{\partial z} & \frac{\partial b}{\partial z} & \frac{\partial c}{\partial z}\\ \frac{\partial d}{\partial u} & \frac{\partial e}{\partial u} & \frac{\partial f}{\partial u} & \frac{\partial d}{\partial x} & \frac{\partial e}{\partial x} & \frac{\partial f}{\partial x}\\ \frac{\partial d}{\partial v} & \frac{\partial e}{\partial v} & \frac{\partial f}{\partial v} & \frac{\partial d}{\partial y} & \frac{\partial e}{\partial y} & \frac{\partial f}{\partial y}\\ \frac{\partial d}{\partial w} & \frac{\partial e}{\partial w} & \frac{\partial f}{\partial w} & \frac{\partial d}{\partial z} & \frac{\partial e}{\partial z} & \frac{\partial f}{\partial z}\end{bmatrix}。$$